Int. J. Solids Structures, 1968, Vol. 4, pp. 323 to 340. Pergamon Press. Printed in Great Britain

THE BUCKLING BEHAVIOUR IN AXIAL COMPRESSION OF SLIGHTLY-CURVED PANELS, INCLUDING THE EFFECT OF SHEAR DEFORMABILITY

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Abstract—The buckling behaviour in axial compression of infinitely-long shallow cylindrical panels with simply-supported edges is analysed, taking shear deformability into account, and the slope of the load–(end strain) curve at the instant of buckling is evaluated; both orthotropic and isotropic materials are considered. The analytical results are of particular relevance to the overall buckling behaviour of panels made from sandwich and other composite materials.

NOTATION

$2a$, $2b$	length and width of panel
a_i, a_{rs}	amplitude of buckling mode i, rs in deflected form
D_x, D_y, D_{xy}	bending rigidities per unit width
D_{x0}	$h^2/4(1-\nu_x\nu_y)F_x$
d_y, d_{xy}	D_y/D_x, D_{xy}/D_x
E	Young's modulus
e_y, e_{xy}	F_y/F_x, F_{xy}/F_x
F_x, F_y, F_{xy}	membrane deformabilities per unit width
$F(\alpha_r, \beta_s)$, $G(\alpha_r, \beta_s)$	defined by equations (32) and (33)
h	thickness
J	defined by equation (34)
k	$\dfrac{(2b)^2}{Rh}\dfrac{(1-\nu_x\nu_y)^{1/2}}{\mu^{1/2}}$
L^i, M^i, N^i_{jk}, P^i_{jkl}	see equation (14)
M_x, M_y, M_{xy}	moments per unit width
N_x, N_y, N_{xy}	membrane forces per unit width
Q_x, Q_y	shear forces per unit width
R	radius of curvature
S_x, S_y	shear deformabilities per unit width, defined by equations (9)
s_x, s_y	$D_xS_x/(2b)^2$, $D_yS_y/(2b)^2$
$S(\theta)$	defined by equations (34), (35) and (36)
w	normal deflection
W	$\dfrac{2}{\pi}\dfrac{(1-\nu_x\nu_y)^{1/2}}{\mu^{1/2}}\dfrac{w}{h}$
x, y	axes shown in Fig. 2
X, Y	x/b, y/b
α_r, β_s, γ	$(2r-1)\dfrac{\lambda\pi}{2}$, $(2s-1)\dfrac{\pi}{2}$, $m\lambda\pi$
ε_x, ε_y, γ_{xy}	membrane strains
ε_0	$-\dfrac{\mu\pi^2}{1-\nu_x\nu_y}\left(\dfrac{h}{2b}\right)^2$

η	mean end load in panel is given by $2\varepsilon_0\eta b/F_x$
η_0	value of η at the instant of buckling
θ_1, θ_2	roots of equation (24)
λ	b/a
μ	$D_x = \mu D_{x0}$
ν_x, ν_y	Poisson's ratios in bending, defined by equations (9)
ν'_x, ν'_y	Poisson's ratios for membrane strains, defined by equations (2)
ϕ_0, ϕ_{rs}	membrane force functions corresponding to modes Ψ_0 and Ψ_{rs}
$\Psi_i(X, Y)$	typical buckling mode
$\Psi_0(X, Y)$	critical buckling mode

1. INTRODUCTION

THE critical buckling modes of elastic panels made from sandwich and other composite materials can usually be divided into two categories. Firstly there are local modes which depend mainly on the detailed construction of the composite material itself, and secondly there are overall modes in which the average stiffness properties and the panel boundary conditions are of primary importance. An adequate theoretical assessment of the elastic properties of composite materials is seldom practicable, so it is usually necessary to perform a series of tests on suitable specimens to investigate the local buckling behaviour and to evaluate the stiffness properties under bending, shearing and in-plane loads. Using these data the critical overall buckling modes may be analysed by treating the panel as an orthotropic plate or shallow shell with finite transverse shear deformability. An analysis of this kind of the overall buckling behaviour of an infinitely-long shallow cylindrical panel in axial compression is given in this Report. The edges are assumed to be intimately attached to longitudinal members which remain in the middle surface of the undeformed panel, and which have sufficient axial rigidity to remain in a state of uniform end strain when buckling takes place; these members have, however, negligible torsional rigidity and negligible bending rigidity in the middle surface of the panel. The analysis is valid provided that the wavelength of the critical overall mode is large compared to the thickness of the composite material. When the wavelength of the overall mode and the thickness are of the same order of magnitude, an adequate theoretical analysis is only possible on those rare occasions when a satisfactory mathematical idealisation can be found to describe the local deformations. Eringen [1] gives an example of such an analysis.

The buckling behaviour in axial compression of cylindrical panels with significant shear deformability has been analysed on the basis of classical small-deflection theory [2] by Leggett and Hopkins [3] and by Stein and Mayers [4]. The results of such analyses must, however, be employed with caution because the extent to which geometrical imperfections influence the buckling behaviour of real panels depends on the stability of the post-buckled behaviour. The slope of the load–(end strain) curve for a flat panel is invariably positive at the instant of buckling and the effect of imperfections on the buckling load is then usually of little importance. The effect of imperfections increases rapidly with the curvature, however, when the shear deformability is zero (see Koiter [5]) due to the decreasing stability of the post-buckled behaviour, which is illustrated in Fig. 1. In general, the buckling load calculated using small deflection theory and assuming no imperfections only gives a useful estimate of the buckling load of a real panel when the post-buckled behaviour is completely stable, as indicated by curve (a). It is often of interest, nevertheless, to know if a panel has the partial stability indicated by curve (b), since buckling during a prescribed variation of the end strain then involves no sudden reduction in the end load.

Koiter [5] gives an analysis of the buckling behaviour in axial compression of an infinitely-long isotropic cylindrical panel with negligible shear deformability, in which the slope of the load–(end strain) curve at the instant of buckling is evaluated exactly. A similar

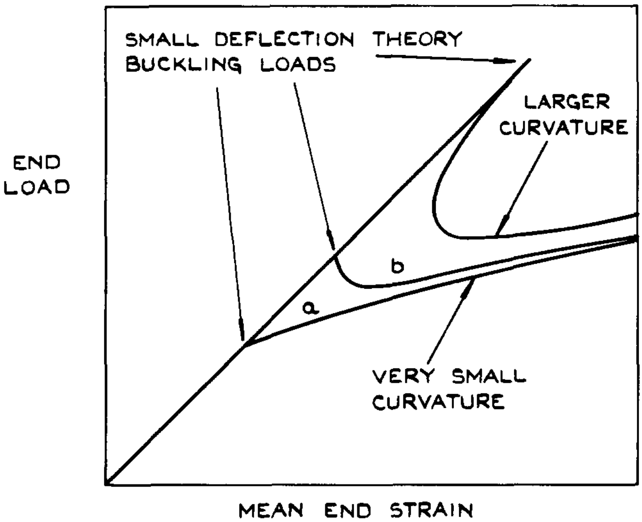

FIG. 1. General form of load–(end strain) curves for perfect panels with zero shear deformability.

analysis is given in this Report for panels with significant shear deformability and with either isotropic or orthotropic properties. Slightly different boundary conditions are employed, however, which lead to considerable simplifications in the analysis without reducing the value of the results. The expressions derived for the buckling load and for the slope of the load–(end strain) curve at the instant of buckling have a relatively simple form and consequently numerical values can easily be computed for any composite material of interest.

The basic equations governing the deformation of orthotropic panels with significant shear deformability are derived briefly in Section 2 in a form similar to that given by Stein and Mayers [6], and the general theory of the stability analysis is outlined in Section 3 where a number of useful relations are presented, which have been derived in a broader context by Thompson [7]. The detailed analysis is given in Section 4 and some illustrative examples are presented in Section 5. The latter demonstrate that the deformation at the instant of buckling can sometimes become more stable with increasing panel curvature when the shear deformability is not zero.

This Report is concerned entirely with the behaviour of panels at the instant of buckling. The elastic post-buckled behaviour of curved panels is discussed by Koiter [5] and by the author [8], who gives an approximate analysis of the deformation after buckling of isotropic panels with negligible shear deformability. The behaviour of orthotropic panels with significant shear deformability would be qualitatively similar if the deformation were purely elastic, but composite materials, which are the main application for these more general properties, frequently deform inelastically as soon as buckling has taken place; a theoretical analysis of the deformation after buckling is then impracticable.

2. BASIC EQUATIONS FOR THE DEFORMATION OF PANELS WITH SIGNIFICANT SHEAR DEFORMABILITY

A brief derivation is given in this section of the basic equations governing the deformation under membrane loading of a cylindrical panel made from a composite material with significant shear deformability. These equations are, in effect, a generalization of the well-known von Karman equations for the finite deflection of plates, to include small cylindrical curvature, orthotropy and shear deformability.

The following overall properties of the composite material, which influence the deformation of the panel, must usually be evaluated by tests on suitable specimens:

(a) The bending and twisting rigidities D_x, D_y and D_{xy} and the corresponding Poisson's ratios ν_x and ν_y. In a homogeneous isotropic material

$$D_x = D_y = Eh^3/12(1-\nu^2), \qquad D_{xy} = Eh^3/12(1+\nu)$$

where h is the thickness.

(b) The membrane deformabilities F_x, F_y and F_{xy}, which can be employed here more conveniently than the membrane rigidities, and the corresponding Poisson's ratios ν'_x and ν'_y. In a homogeneous isotropic material

$$F_x = F_y = 1/Eh, \qquad F_{xy} = 2(1+\nu)/Eh.$$

(c) The shear deformabilities S_x and S_y.

These twelve material properties are defined by the (membrane force)–strain and moment–(shear force) curvature relationships given in Sections 2.1 and 2.2. Only ten independent properties are involved, however, because the values of ν_x and ν_y and of ν'_x and ν'_y are linked by the reciprocal theorem.

The x and y axes are chosen to lie in the middle surface of a curved panel of radius R and thickness h as shown in Fig. 2. The x axis is parallel to the generators and the edges are

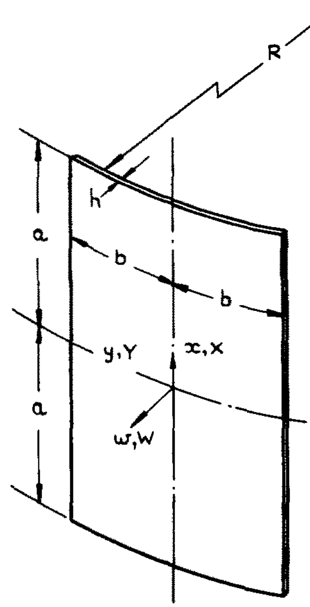

FIG. 2. Axes and dimensions.

defined by $x = \pm a$ and $y = \pm b$. In the subsequent analysis the limiting case will be considered when the length $2a$ tends to infinity.

It is convenient to relate the bending properties of the composite material to those of a reference sandwich material with the same membrane deformability, the same thickness and the same Poisson's ratio properties in bending, but with all the membrane-stress-carrying capability concentrated in covers which are very thin compared with the total thickness. The bending rigidity of such a material about axes parallel to the y axis is given by

$$D_{x0} = h^2/4(1-\nu_x\nu_y)F_x.$$

The corresponding bending rigidity of the actual material may then be expressed as

$$D_x = \mu D_{x0}.$$

The parameter μ, which depends on the form of construction, has the value 1/3 when the panel is homogeneous.

The subsequent analysis is expressed for simplicity in terms of the following non-dimensional notation:

$$X = x/b, \qquad Y = y/b,$$

$$k = \frac{(2b)^2}{Rh}\frac{(1-\nu_x\nu_y)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}, \qquad W = \frac{2}{\pi}\frac{(1-\nu_x\nu_y)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}\frac{w}{h}$$

where w is the radial deflection. It is useful to note that the eccentricity δ of the panel centre-line relative to the plane containing the edges is given by

$$\frac{\delta}{h} = \frac{(2b)^2}{8Rh} = \frac{\mu^{\frac{1}{2}}k}{8(1-\nu_x\nu_y)^{\frac{1}{2}}}.$$

2.1 *Membrane force function*

The condition that the membrane strains ε_x, ε_y and γ_{xy} are compatible is given by

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y} = \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2}-\frac{1}{R}\right),$$

which may be re-expressed in the following non-dimensional form:

$$\frac{\partial^2\varepsilon_x}{\partial Y^2}+\frac{\partial^2\varepsilon_y}{\partial X^2}-\frac{\partial^2\gamma_{xy}}{\partial X\partial Y} = \frac{\mu\pi^2}{1-\nu_x\nu_y}\left(\frac{h}{2b}\right)^2\left\{\left(\frac{\partial^2 W}{\partial X\partial Y}\right)^2-\frac{\partial^2 W}{\partial X^2}\left(\frac{\partial^2 W}{\partial Y^2}-\frac{k}{2\pi}\right)\right\}. \tag{1}$$

These strains are related to the corresponding membrane forces N_x, N_y and N_{xy} by the expressions

$$\left.\begin{aligned} \varepsilon_x = F_xN_x-\nu_y'F_yN_y, \qquad \varepsilon_y = F_yN_y-\nu_x'F_xN_x,\\ \gamma_{xy} = F_{xy}N_{xy} \end{aligned}\right\} \tag{2}$$

where the Poisson's ratios ν_x' and ν_y' are linked by the reciprocal relation

$$\nu_x'F_x = \nu_y'F_y. \tag{3}$$

The membrane forces are in equilibrium so they may be expressed in terms of a function ϕ such that

$$N_x, N_y, N_{xy} = \frac{\varepsilon_0}{F_x}\left(\frac{\partial^2\phi}{\partial Y^2}, \frac{\partial^2\phi}{\partial X^2}, -\frac{\partial^2\phi}{\partial X\partial Y}\right) \tag{4}$$

where the factor ε_0/F_x is introduced for convenience, and where

$$\varepsilon_0 = -\frac{\mu\pi^2}{1-\nu_x\nu_y}\left(\frac{h}{2b}\right)^2. \tag{5}$$

In the special case when the bending behaviour of the composite material is isotropic and the shear deformability is negligible, this reference strain is the end strain at which a flat, square, simply-supported panel would buckle in axial compression.

Substituting equations (4) in equations (2) and then substituting equations (2) in equation (1), we obtain

$$e_y\frac{\partial^4\phi}{\partial X^4}+(e_{xy}-\nu_x'-e_y\nu_y')\frac{\partial^4\phi}{\partial X^2\partial Y^2}+\frac{\partial^4\phi}{\partial Y^4} = -\left(\frac{\partial^2 W}{\partial X\partial Y}\right)^2+\frac{\partial^2 W}{\partial X^2}\left(\frac{\partial^2 W}{\partial Y^2}-\frac{k}{2\pi}\right) \tag{6}$$

where

$$e_y = F_y/F_x, \qquad e_{xy} = F_{xy}/F_x.$$

2.2 *Radial equilibrium equation*

Consider an element of the panel subjected to moments M_x, M_y and M_{xy}, shear forces Q_x, Q_y and membrane forces N_x, N_y and N_{xy}, as shown in Fig. 3. Taking moments about the x and y axes and resolving forces normal to the element, the following three equations are obtained:

$$\left.\begin{aligned}\frac{\partial M_x}{\partial x}-\frac{\partial M_{xy}}{\partial y}-Q_x &= 0,\\ \frac{\partial M_y}{\partial y}-\frac{\partial M_{xy}}{\partial x}-Q_y &= 0,\end{aligned}\right\} \tag{7}$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+N_x\frac{\partial^2 w}{\partial x^2}+N_y\left(\frac{\partial^2 w}{\partial y^2}-\frac{1}{R}\right)+2N_{xy}\frac{\partial^2 w}{\partial x\partial y} = 0. \tag{8}$$

The relationships between the normal deflection and the moments and shear forces may be written in the form,

$$\left.\begin{aligned}M_x &= -D_x\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}-S_xQ_x\right)+\nu_y\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}-S_yQ_y\right)\right],\\ M_y &= -D_y\left[\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}-S_yQ_y\right)+\nu_x\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}-S_xQ_x\right)\right],\\ M_{xy} &= \frac{D_{xy}}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}-S_yQ_y\right)+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}-S_xQ_x\right)\right]\end{aligned}\right\} \tag{9}$$

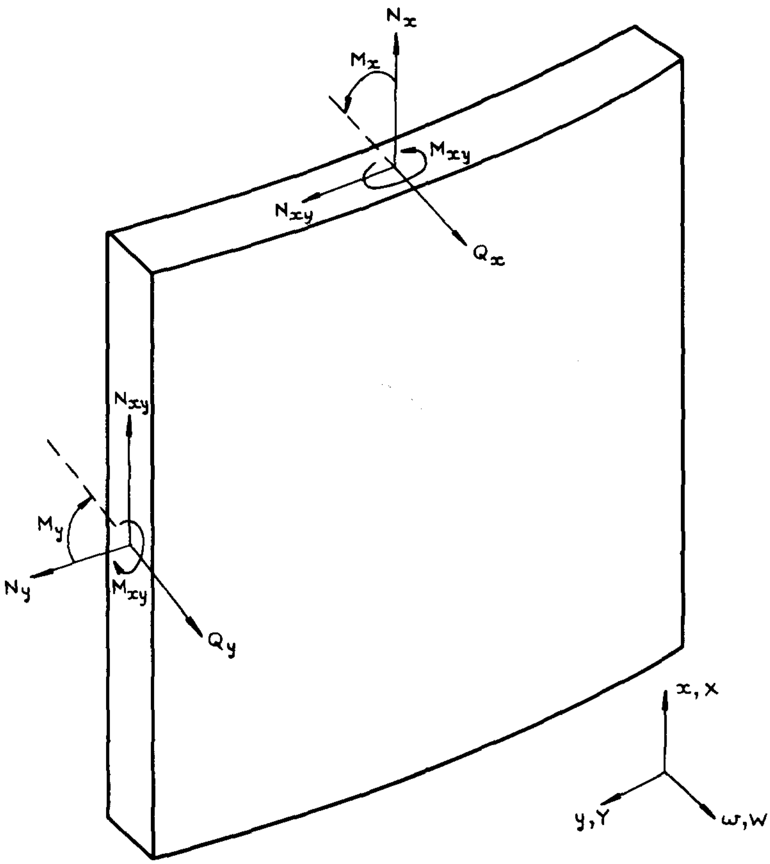

FIG. 3. Forces on a typical element.

where the Poisson's ratios ν_x and ν_y are linked by the reciprocal relation

$$\nu_x D_y = \nu_y D_x.$$

Substituting equations (9) in equations (7) and employing the resulting equations to eliminate Q_x and Q_y from equation (8), the following equation is obtained if the membrane forces are expressed in terms of the force function ϕ:

$$\frac{\partial^4 W}{\partial X^4}+(2d_{xy}+\nu_x d_y+\nu_y)\frac{\partial^4 W}{\partial X^2 \partial Y^2}+d_y\frac{\partial^4 W}{\partial Y^4}$$

$$-2\left[\frac{d_{xy}}{d_y}s_y\frac{\partial^6 W}{\partial X^6}+\left\{2s_y(1-\nu_x\nu_y)-d_{xy}s_y\left(\nu_x+\frac{\nu_y}{d_y}\right)+d_{xy}s_x\right\}\frac{\partial^6 W}{\partial X^4 \partial Y^2}\right.$$

$$\left.+\{2d_y s_x(1-\nu_x\nu_y)-d_{xy}s_x(\nu_x d_y+\nu_y)+d_{xy}s_y\}\frac{\partial^6 W}{\partial X^2 \partial Y^4}+d_{xy}d_y s_x\frac{\partial^6 W}{\partial Y^6}\right]$$

$$+\pi^2\left\{\left(1-4s_x\frac{\partial^2}{\partial X^2}-2d_{xy}s_x\frac{\partial^2}{\partial Y^2}\right)\left(1-2\frac{d_{xy}}{d_y}s_y\frac{\partial^2}{\partial X^2}-4s_y\frac{\partial^2}{\partial Y^2}\right)\right.$$

$$-4s_x s_y(2\nu_y+d_{xy})\left(2\nu_x+\frac{d_{xy}}{d_y}\right)\frac{\partial^4}{\partial X^2\partial Y^2}\bigg\}\bigg\{\frac{\partial^2\phi}{\partial Y^2}\frac{\partial^2 W}{\partial X^2}$$

$$+\frac{\partial^2\phi}{\partial X^2}\left(\frac{\partial^2 W}{\partial Y^2}-\frac{k}{2\pi}\right)-2\frac{\partial^2\phi}{\partial X\partial Y}\frac{\partial^2 W}{\partial X\partial Y}\bigg\} = 0 \tag{10}$$

where

$$d_y = D_y/D_x, \qquad d_{xy} = D_{xy}/D_x,$$
$$s_x = D_x S_x/(2b)^2, \qquad s_y = D_y S_y/(2b)^2.$$

3. BASIC ANALYSIS OF THE BUCKLING BEHAVIOUR

Let the deflection of the panel be expressed in the form

$$W = \sum_{i=0}^{\infty} a_i \Phi_i(X, Y) \tag{11}$$

where the functions $\Phi_i(X, Y)$ represent all the possible buckling modes of the panel under the applied loading. An infinite set of simultaneous equations for the amplitudes a_i may be derived using the principle of virtual displacements, which may here be expressed in the form [9, 10]

$$\int_{-1}^{+1}\int_{-a/b}^{+a/b} U\,\delta W\,\mathrm{d}X\,\mathrm{d}Y = 0 \tag{12}$$

where U represents the left hand side of equation (10), and where δW is an infinitesimal arbitrary virtual displacement satisfying all the boundary conditions. Substituting virtual displacements given by

$$\delta W_i = \delta a_i \Phi_i(X, Y) \tag{13}$$

in equation (12), and using equation (6) to express the membrane force function ϕ in terms of the displacement functions $\Phi_i(X, Y)$, the following infinite set of simultaneous equations is obtained:

$$(L^i+\eta M^i)a_i+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} N^i_{jk}a_j a_k+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=o}^{\infty} P^i_{jkl}a_j a_k a_l = 0 \tag{14}$$

where $2\varepsilon_0\eta b/F_x$ is the mean end load in the panel and where L^i, M^i, N^i_{jk} and P^i_{jkl} are constants.

If now the function $\Phi_0(X, Y)$ is chosen to represent the critical mode, buckling will occur due to a monotonically increasing end load when the parameter η reaches the value

$$\eta_0 = -L^0/M^0. \tag{15}$$

The following expressions, which are derived in more general terms and in greater detail by Thompson [7], may be deduced from equations (14):

$$\operatorname*{Lt}_{\eta\to\eta_0} \frac{a_i}{a_0^2} = -\frac{N_{00}^i}{L^i+\eta_0 M^i}, \qquad i \neq 0, \tag{16}$$

$$\operatorname*{Lt}_{\eta\to\eta_0} \frac{d\eta}{da_0} = -\frac{N_{00}^0}{M^0}. \tag{17}$$

When the sign of the amplitude a_0 is arbitrary, $N_{00}^0 = 0$ and

$$\operatorname*{Lt}_{\eta\to\eta_0} \frac{d^2\eta}{da_0^2} = -\frac{2}{M^0}\left\{P_{000}^0 - \sum_{i=1}^{\infty} \frac{2(N_{00}^i)^2}{L^i+\eta_0 M^i}\right\}. \tag{18}$$

4. THE BUCKLING OF SIMPLY-SUPPORTED PANELS

The buckling behaviour in axial compression of an infinitely-long shallow cylindrical panel with significant shear deformability is now analysed, using the theory outlined in Section 3. A panel of finite length is considered first with a somewhat artificial combination of boundary conditions at the ends; the results obtained for this panel are then used to deduce the buckling behaviour of a corresponding infinitely-long panel.

4.1 *Panel of finite length*

Consider a cylindrical panel bounded along the straight edges by members which are constrained to stay in the middle surface of the unbuckled panel, and which are stiff enough to remain in a state of uniform end strain when the panel buckles. If these members have negligible torsional rigidity and negligible bending rigidity parallel to the middle surface of the panel, the boundary conditions on the edges $y = \pm b$ are given by

$$w = 0, \quad M_y = 0, \quad N_y = 0, \quad Q_x = 0 \quad \text{and } \varepsilon_x = \text{const.} \tag{19}$$

Substituting these boundary conditions in equation (8) and in the second of equations (9), and eliminating Q_y, we obtain

$$\frac{\partial^2 w}{\partial y^2} = -2S_y N_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

which may be rewritten in the form

$$\frac{\partial^2 W}{\partial Y^2} = 2S_y \frac{\varepsilon_0}{F_x} \frac{\partial^2 \phi}{\partial X \partial Y} \frac{\partial^2 W}{\partial X \partial Y}. \tag{20}$$

When the shear deformability is negligible this equation reduces to

$$\partial^2 W/\partial Y^2 = 0. \tag{21}$$

Consider now the expressions for ϕ which are given in equations (23), (25) and (27) of the subsequent analysis. It follows from these equations and from equation (16) that $\partial^2\phi/\partial X \partial Y$ becomes a function of a_0 (or a_0^2 if $k = 0$) in the limit as η tends to η_0. The value of $\partial^2 W/\partial Y^2$ on $Y = \pm 1$ is then a function of a_0^2 (or a_0^3 is $k = 0$) by virtue of equation (20). This value of $\partial^2 W/\partial Y^2$ is negligibly small compared to the corresponding limiting value on the panel centre-line, which is a function of a_0, so the simplified boundary condition (21) may be employed in this analysis even when the shear deformability is significant.

The ends of this hypothetical panel are subjected to a system of constraints which act in such a way that all the buckling modes in axial compression are also buckling modes of a similar panel of infinite length under the same loading; they are thus specified by a deflected form which is sinusoidal both axially and circumferentially. It may readily be verified that the amplitude of the critical buckling mode of the relatively shallow panels considered here varies across the width as $\cos \pi Y/2$. Suppose, now, that the properties of the panel are such that the deflected form in the critical mode is antisymmetric about the Y axis and is thus of the form

$$W_0 = a_0 \sin \gamma X \cos \beta_1 Y \tag{22}$$

where

$$\gamma = m\lambda\pi, \qquad \beta_n = (2n-1)\pi/2, \qquad \lambda = b/a$$

and where m is an integer which is chosen to make the buckling load a minimum. The corresponding membrane force function may be deduced from equations (6) and (19). When the membrane properties are orthotropic this function is given by

$$\phi_0 = \frac{\eta Y^2}{2} + \frac{k}{2\pi}\frac{a_0\gamma^2 \sin \gamma X \cos \beta_1 Y}{e_y\gamma^4 + (e_{xy} - v'_x - e_y v'_y)\gamma^2\beta_1^2 + \beta_1^4}$$
$$-\frac{a_0^2}{32}\left[\frac{1}{e_y}\frac{\beta_1^2}{\gamma^2}\cos 2\gamma X\left\{1 + \frac{1}{\theta_1^2 - \theta_2^2}\left(\theta_2^2\frac{\cosh \theta_1 Y}{\cosh \theta_1} - \theta_1^2\frac{\cosh \theta_2 Y}{\cosh \theta_2}\right)\right\} - \frac{\gamma^2}{\beta_1^2}\cos 2\beta_1 Y\right] \tag{23}$$

where θ_1 and θ_2 are the positive roots of the equation

$$\theta^4 - 4(e_{xy} - v'_x - e_y v'_y)\gamma^2\theta^2 + 16 e_y\gamma^4 = 0. \tag{24}$$

When the membrane properties are isotropic this force function is given by

$$\phi_0 = \frac{\eta Y^2}{2} + \frac{k}{2\pi}\frac{a_0\gamma^2 \sin \gamma X \cos \beta_1 Y}{(\gamma^2 + \beta_1^2)^2}$$
$$-\frac{a_0^2}{32}\left[\frac{\beta_1^2}{\gamma^2}\cos 2\gamma X\left\{1 - \frac{1}{\cosh^2 2\gamma}(\overline{\cosh 2\gamma + \gamma \sinh 2\gamma} \cosh 2\gamma Y - \gamma Y \cosh 2\gamma \sinh 2\gamma Y)\right\}\right.$$
$$\left.-\frac{\gamma^2}{\beta_1^2}\cos 2\beta_1 Y\right]. \tag{25}$$

Since the critical mode is antisymmetric about the Y axis, the sign of the amplitude is arbitrary at the instant of buckling and consequently

$$\underset{\eta \to \eta_0}{\mathrm{Lt}} \frac{\mathrm{d}\eta}{\mathrm{d}a_0} = 0.$$

It can readily be demonstrated that all the coefficients of a_0^2 in equation (10) are symmetrical with respect to both the X and Y axes. Consequently the corresponding terms N_{00}^i in equations (14) are only non-zero when the higher buckling modes are also doubly-symmetrical. It follows therefore from equation (16) that the behaviour of the panel at the instant of buckling may be analysed exactly by expressing the deflection in the following form:

$$W = a_0 \sin \gamma X \cos \beta_1 Y + \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} a_{rs} \cos \alpha_r X \cos \beta_s Y \tag{26}$$

where

$$\alpha_r = (2r-1)\lambda \pi/2$$

and where the double suffix rs is used, for convenience, to replace the single suffix i in the basic theory given in Section 3. The membrane force function corresponding to a typical doubly-symmetrical mode may here be represented by

$$\phi_{rs} = \frac{k}{2\pi} \frac{a_{rs}\alpha_r^2 \cos \alpha_r X \cos \beta_s Y}{e_y\alpha_r^4 + (e_{xy} - v'_x - e_y v'_y)\alpha_r^2\beta_s^2 + \beta_s^4}; \tag{27}$$

further terms in the stress function, which involve a_{rs}^2 and products of the amplitudes of different modes, do not enter the analysis.

Substituting equations (23), (25), (26) and (27) in equation (14), and then substituting the relevant terms of the latter equation in equations (15) and (18), we obtain the following expressions:

$$\eta_0 = \frac{1}{\pi^2\gamma^2 G(\gamma, \beta_1)}\left\{F(\gamma, \beta_1) + \frac{k^2}{4}\frac{\gamma^4 G(\gamma, \beta_1)}{e_y\gamma^4 + (e_{xy} - v'_x - e_y v'_y)\gamma^2\beta_1^2 + \beta_1^4}\right\}, \tag{28}$$

$$\underset{\eta \to \eta_0}{\mathrm{Lt}} \frac{\mathrm{d}^2\eta}{\mathrm{d}a_0^2} = \frac{1}{8\gamma^2}\left[\gamma^4 + \beta_1^4\left\{\frac{1}{e_y} + \frac{1}{G(\gamma, \beta_1)}\left(\frac{J}{2e_y} - k^2 V\right)\right\}\right] \tag{29}$$

where

$$V = 32\lambda^2 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} R_{rs}, \tag{30}$$

$$R_{rs} = \frac{\left[\dfrac{2\gamma^4}{(\gamma^2+\beta_1^2)^2}\left\{\dfrac{\beta_s G(0, 2\beta_1)}{\alpha_r(4\beta_1^2 - \beta_s^2)} + \dfrac{\alpha_r G(2\gamma, 0)}{\beta_s(4\gamma^2 - \alpha_r^2)}\right\} + \dfrac{\alpha_r G(2\gamma, 0)}{4\beta_s(4\gamma^2 - \alpha_r^2)}\right]^2}{F(\alpha_r, \beta_s) + \dfrac{k^2}{4}\dfrac{\alpha_r^4 G(\alpha_r, \beta_s)}{e_y\alpha_r^4 + (e_{xy} - v'_x - e_y v'_y)\alpha_r^2\beta_s^2 + \beta_s^4} - \pi^2\alpha_r^2\eta_0 G(\alpha_r, \beta_s)} \tag{31}$$

and where

$$\begin{aligned} F(\alpha_r, \beta_s) = {} & \alpha_r^4 + (2d_{xy} + v_x d_y + v_y)\alpha_r^2\beta_s^2 + d_y\beta_s^4 \\ & + 2\left[\frac{d_{xy}}{d_y}s_y\alpha_r^6 + \left\{2s_y(1 - v_x v_y) - d_{xy}s_y\left(v_x + \frac{v_y}{d_y}\right) + d_{xy}s_x\right\}\alpha_r^4\beta_s^2\right. \\ & + \{2d_y s_x(1 - v_x v_y) - d_{xy}s_x(v_x d_y + v_y) + d_{xy}s_y\}\alpha_r^2\beta_s^4 \\ & \left. + d_{xy}d_y s_x\beta_s^6\right], \end{aligned} \tag{32}$$

$$\begin{aligned} G(\alpha_r, \beta_s) = {} & (1 + 4s_x\alpha_r^2 + 2d_{xy}s_x\beta_s^2)\left(1 + 2\frac{d_{xy}}{d_y}s_y\alpha_r^2 + 4s_y\beta_s^2\right) \\ & - 4s_x s_y(2v_y + d_{xy})\left(2v_x + \frac{d_{xy}}{d_y}\right)\alpha_r^2\beta_s^2. \end{aligned} \tag{33}$$

When the membrane properties are orthotropic

$$J = \frac{2}{\theta_1\theta_2(\theta_1^2-\theta_2^2)}\{\theta_2^3 S(\theta_1)\tanh\theta_1 - \theta_1^3 S(\theta_2)\tanh\theta_2\}, \tag{34}$$

where

$$S(\theta) = H_1 + (\beta_1^2 - 2\theta^2)H_2 + (\beta_1^4 - 8\beta_1^2\theta^2 + 3\theta^4)H_3 \tag{35}$$

and

$$H_1 = 1 + 2\gamma^2\left(\frac{d_{xy}}{d_y}s_y + 2s_x\right) + 8\frac{d_{xy}}{d_y}s_x s_y \gamma^4,$$

$$H_2 = 2\left[4\gamma^2 s_x s_y\left\{2(1-v_x v_y) - d_{xy}\left(\frac{v_y}{d_y} + v_x\right)\right\} + d_{xy}s_x + 2s_y\right],$$

$$H_3 = 8d_{xy}s_x s_y.$$

When the membrane properties are isotropic,

$$J = S(2\gamma)\operatorname{sech}^2 2\gamma - \frac{\tanh 2\gamma}{2\gamma}\{3H_1 + (3\beta_1^2 - 8\gamma^2)H_2 + (3\beta_1^4 - 32\beta_1^2\gamma^2 - 48\gamma^4)H_3\}. \tag{36}$$

The mean end strain in an element of the panel parallel to the X axis is here denoted by the product $\varepsilon\varepsilon_0$ where

$$\varepsilon = \frac{\lambda}{2}\int_{-1/\lambda}^{1/\lambda}\left\{\frac{\partial^2\phi}{\partial Y^2} - v\frac{\partial^2\phi}{\partial X^2} + \frac{1}{2}\left(\frac{\partial W}{\partial X}\right)^2\right\}\mathrm{d}X. \tag{37}$$

The mean end strain immediately after buckling may be evaluated by substituting equations (23), (25), (26) and (27) in equation (37), and by noting that a_0^2 and a_{rs} are of the same order of magnitude in the limit as η tends to η_0 (see equation (16)). The following expression is then obtained:

$$\varepsilon = \eta + \frac{a_0^2\gamma^2}{8} + \frac{k\lambda}{4\pi}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} a_{rs}(-1)^r \frac{\alpha_r^2(\beta_s^2 - v\alpha_r^2)\cos\beta_s Y}{e_y\alpha_r^4 + (e_{xy} - v_x' - e_y v_y')\alpha_r^2\beta_s^2 + \beta_s^4}. \tag{38}$$

4.2 *Panel of infinite length*

The foregoing analysis of the buckling of a panel of finite length is of little practical interest as it stands, since the deformation in the buckled state corresponds neither to a uniformly distributed load on the ends nor to a uniform mean end strain across the width. It is evident from equation (38), however, that the mean end strain is uniform across the width in the limit when the length tends to infinity ($\lambda \to 0$). The results obtained for this panel of finite length may therefore be used to deduce the behaviour of the corresponding infinitely long panel in uniform axial compression, provided that this limiting case is not influenced by the implicit assumption that the critical buckling mode is antisymmetric about the Y axis. Now the amplitude a_0 of the critical buckling mode is of arbitrary sign when the buckling mode is antisymmetric [i.e. $\operatorname{Lt}_{\eta\to\eta_0}(\mathrm{d}\eta/\mathrm{d}a_0) = 0$], but it has a preferred sign when the mode is symmetrical and the length of the panel is finite [i.e. $\operatorname{Lt}_{\eta\to\eta_0}(\mathrm{d}\eta/\mathrm{d}a_0 \neq 0$ when the curvature is non-zero]. Consequently the relationship between the load parameter η and the strain parameter ε at the instant of buckling for a panel of finite length can have two distinct forms, depending on the nature of the critical buckling mode. When, however, the length of the panel is infinite, the sign of the amplitude of the symmetrical critical mode is also arbitrary and it can then be demonstrated that the same result is obtained irrespective of whether the critical mode is assumed to be symmetrical or antisymmetrical.

4.2.1 *The critical buckling load.* The buckling load of the infinitely-long panel in axial compression may be evaluated directly by substitution in equation (28) if the wavelength of the buckles along the X axis is specified; the minimum buckling load may thus be deduced by varying this wavelength. It is usually easiest to perform this minimisation process numerically. When, however, the shear deformability is negligible and the bending and membrane properties are such that

$$e_y d_y = 1, \tag{39}$$

the minimum buckling load occurs when

$$\gamma = \frac{\pi}{2} d_y^{\frac{1}{4}} \tag{40}$$

provided that

$$k \leq \frac{\pi^2}{2}\{2+(e_{xy}-v_x'-e_y v_y')e_y^{-\frac{1}{2}}\}. \tag{41}$$

Thus the axial and circumferential wavelengths of the buckles are equal when the panel is isotropic if

$$k \leq 2\pi^2.$$

This result for the isotropic panel has previously been given by Timoshenko and Gere [2].

4.2.2 *The slope of the load–(end strain) curve at the instant of buckling.* The critical buckling mode may be substituted in equation (29) to evaluate $d^2\eta/da_0^2$ at the instant of buckling. The parameter λ is zero, however, in the limit when the panel is infinitely long and consequently the term V in this equation, which is defined by equations (30) and (31), can only be non-zero when α_r assumes the value 0, γ or 2γ. This term may be evaluated with the aid of the following relationships which are obtained by expressing α_r and γ in terms of r and m:

$$\left.\begin{aligned} \sum_{r=1}^{\infty} \frac{\lambda^2}{\alpha_r^2} &= \frac{4}{\pi^2}\sum_{r=1}^{\infty}\left(\frac{1}{2r-1}\right)^2 = \frac{1}{2}, \\ \operatorname*{Lt}_{\lambda\to 0} \sum_{r=1}^{\infty}\left(\frac{\lambda^2}{\alpha_r^2-\gamma^2}\right) &= 0, \end{aligned}\right\} \tag{42}$$

$$\operatorname*{Lt}_{\lambda\to 0} \sum_{r=1}^{\infty}\left(\frac{\lambda\alpha_r}{4\gamma^2-\alpha_r^2}\right)^2 = \frac{1}{\pi^2}\sum_{r=1}^{\infty}\left(\frac{1}{4m-2r+1}\right)^2.$$

Since 2γ is finite, the terms of this series are only non-zero in the limit as $\lambda \to 0$ when both m and r are infinite. Consequently

$$\operatorname*{Lt}_{\lambda\to 0} \sum_{r=1}^{\infty}\left(\frac{1}{4m-2r+1}\right)^2 = 2\sum_{r=1}^{\infty}\left(\frac{1}{2r-1}\right)^2 = \frac{\pi^2}{4}$$

and

$$\operatorname*{Lt}_{\lambda\to 0}\sum_{r=1}^{\infty}\left(\frac{\lambda\alpha_r}{4\gamma^2-\alpha_r^2}\right)^2=\frac{1}{4}. \tag{43}$$

Substituting equations (42) and (43) in equations (30) and (31), we obtain

$$\begin{aligned}V = 32\sum_{s=1}^{\infty}\Bigg[&\left\{\left(\frac{2\gamma^4}{(\gamma^2+\beta_1^2)^2}+\frac{1}{4}\right)\frac{G(2\gamma,0)}{2\beta_s}\right\}^2\\ &\times\left\{F(2\gamma,\beta_s)+\frac{4k^2\gamma^4G(2\gamma,\beta_s)}{16e_y\gamma^4+4(e_{xy}-\nu_x'-e_y\nu_y')\gamma^2\beta_s^2+\beta_s^4}-4\pi^2\gamma^2\eta_0G(2\gamma,\beta_s)\right\}^{-1}\\ &+2\left\{\frac{\gamma^4\beta_sG(0,2\beta_s)}{(\gamma^2+\beta_1^2)^2(4\beta_1^2-\beta_s^2)}\right\}^2\{d_y\beta_s^4(1+2d_{xy}s_x\beta_s^2)\}^{-1}\Bigg].\end{aligned} \tag{44}$$

The mean end strain in the infinitely-long panel immediately after buckling is obtained by putting $\lambda = 0$ in equation (38). Differentiating this equation with respect to η and noting that

$$\operatorname*{Lt}_{\eta\to\eta_0}\frac{d\eta}{da_0}=0,$$

the following expression may be deduced for the slope of the load–(mean end strain) curve at the instant of buckling:

$$\operatorname*{Lt}_{\eta\to\eta_0}\frac{d\eta}{d\varepsilon}=\operatorname*{Lt}_{\eta\to\eta_0}\left\{\left(\frac{d^2\eta}{da_0^2}+\frac{\gamma^2}{4}\right)^{-1}\frac{d^2\eta}{da_0^2}\right\}. \tag{45}$$

5. ILLUSTRATIVE EXAMPLES

The theory developed in the preceding sections is now applied to two specific classes of infinitely-long panels.

5.1 *Isotropic panels*

As a first numerical example we consider panels whose properties are isotropic in the sense that

$$\begin{aligned}&e_y = 1, && e_{xy} = 2(1+\nu'), && \nu' = \nu_x' = \nu_y'\\ &d_y = 1, && d_{xy} = 1-\nu, && \nu = \nu_x = \nu_y,\end{aligned}$$
$$s = s_x = s_y.$$

It is assumed in the computations that

$$\nu = \nu' = 0{\cdot}3.$$

Figure 4 shows the variation of the buckling load parameter η_0 and the wavelength parameter $\pi/2\gamma$ with the curvature parameter k and the shear deformability parameter s. It follows from equations (39) and (41) that the wavelength of buckling is independent of the

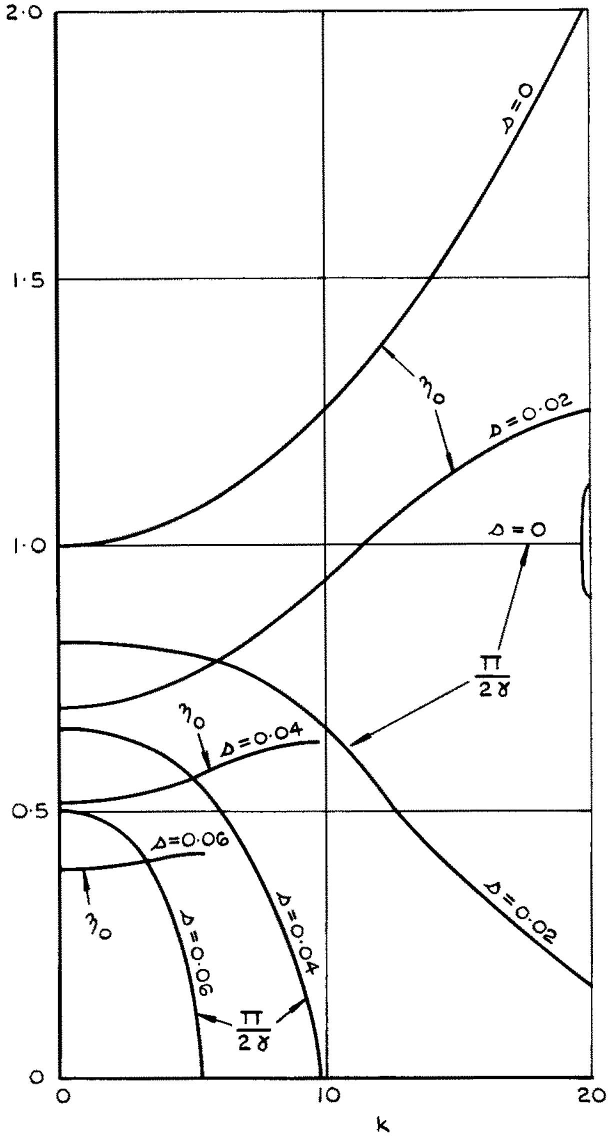

FIG. 4. Buckling stresses and wavelengths for isotropic panels.

curvature when $s = 0$ and $k \leq 2\pi^2$. The wavelength of the initial buckles decreases, however, with increasing curvature and/or shear deformability when the latter is significant. Figure 5 illustrates that shear deformability tends to have a destabilising effect on the post-buckled behaviour when k and s are both small; it also shows, however, that this trend is not maintained when the wavelength of buckling is very short. Consider, for example, the results obtained when $s = 0{\cdot}06$. The stability of the load–(end strain) curve at the instant of buckling decreases with increasing curvature when k is less than about 4. When, however, k is greater than 4, the stability of the load–(end strain) curve increases with the curvature until

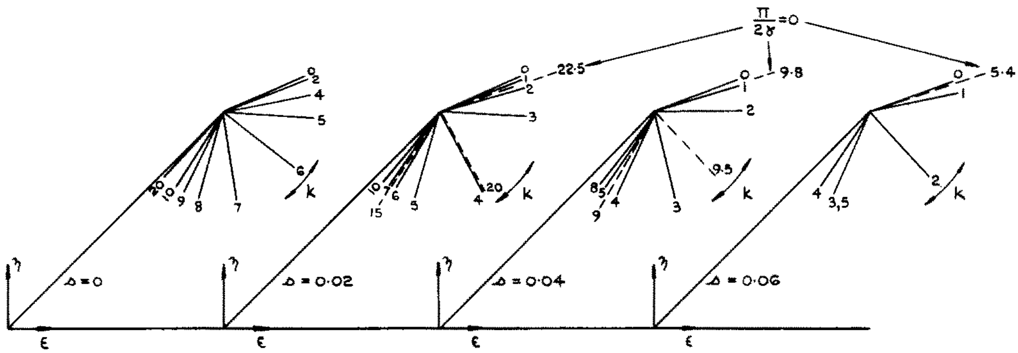

Fig. 5. Variation with k and s of the initial slope of the η vs. ε curve when buckling occurs—isotropic panels.

the wavelength becomes zero when k is about 5·4. It follows immediately from equation (29) that the slope of the load–(end strain) curve at the instant of buckling is independent of the curvature and the shear deformability whenever the properties of the panel are such that the wavelength of buckling is zero. This limiting case is largely of academic interest, however, because the analysis is only applicable to composite materials when the wavelength is large compared to the thickness.

It is interesting to compare the results obtained here for a panel with negligible shear deformability to those given by Koiter [5] for a set of identical panels forming a complete cylindrical shell. These panels are separated by stringers whose properties are such that the buckling load of a typical panel is identical with that obtained in the present analysis. However, the post-buckled stiffness is larger because no slope discontinuity is permitted between the normal deflections of adjacent panels. The results of the two analyses are compared in the following table:

Tangent to η vs. ε curve at the instant of buckling	Value of k	
	Koiter	Present analysis
Horizontal	8·1	4·8
Vertical downwards	11·7	7·2

5.2 *Sandwich panels with corrugated cores*

Sandwich panels with cores consisting of heavy corrugations parallel to the X axis are chosen as a second numerical example. The longitudinal shear deformability s_x of such panels is very small and is here assumed to be negligible; numerical results are given for panels with the following membrane and bending properties:

$$e_y = 2, \qquad e_{xy} = 5{\cdot}2, \qquad \nu'_x = 0{\cdot}3,$$
$$d_y = 0{\cdot}5, \qquad d_{xy} = 0{\cdot}35, \qquad \nu_x = 0{\cdot}3.$$

The variation of the buckling load parameter η_0 and the wavelength parameter $\pi/2\gamma$ with the curvature parameter k and the shear deformability parameter s_y is shown in Fig. 6.

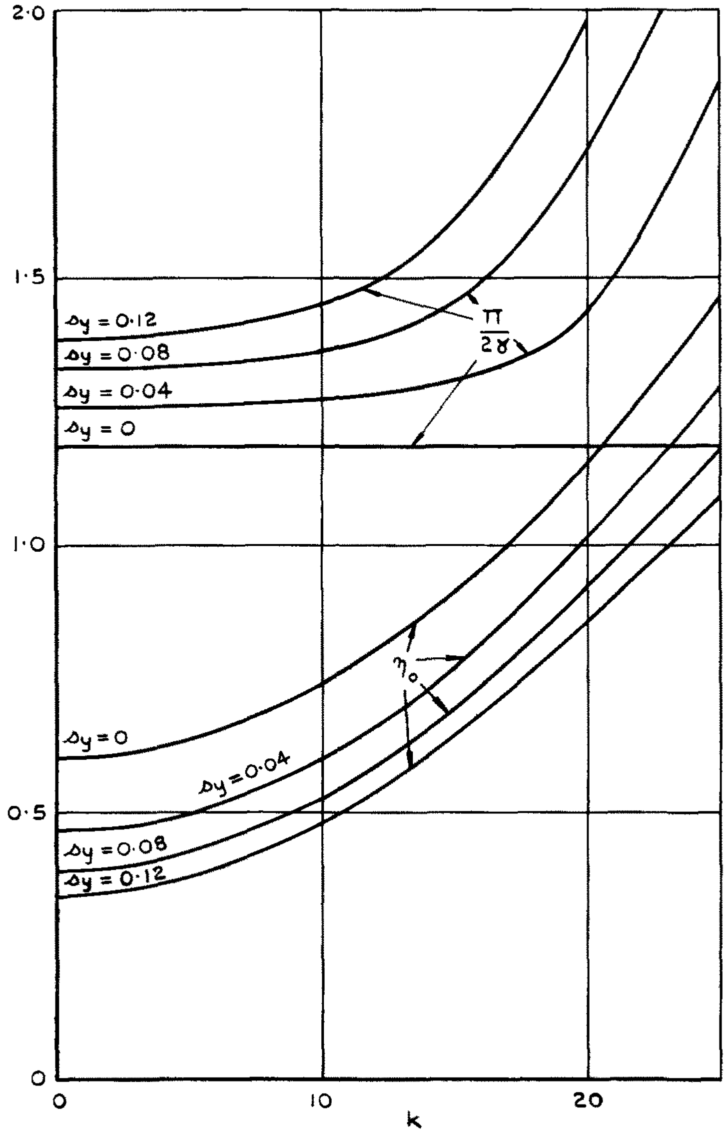

FIG. 6. Buckling stresses and wavelengths — $s_x = 0$.

The product $e_y d_y$ is here unity, so it again follows from equation (39) that the wavelength of buckling is independent of the curvature when $s_y = 0$ and when k is less than or equal to the value specified by equation (41). The wavelength of the initial buckles increases, however, with the curvature and/or the shear deformability when the latter is significant. It is illustrated in Fig. 7 that shear deformability tends to have a destabilising effect on the post-buckled behaviour when the panel is curved. Shear deformability does, nevertheless, slightly increase the slope of the post-buckled load–(end strain) curve of flat panels at the

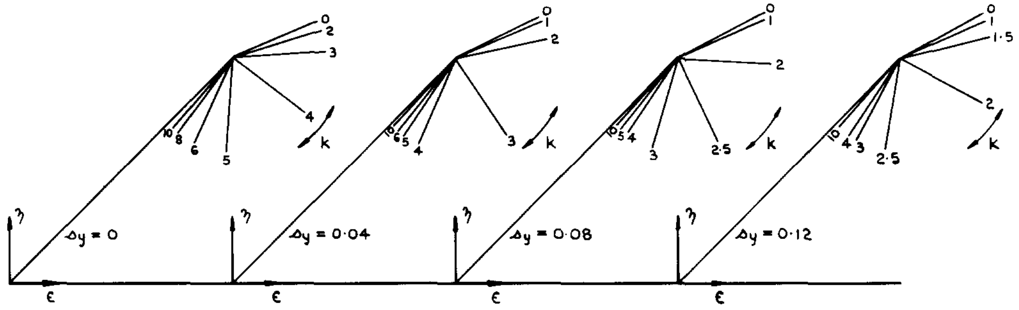

FIG. 7. Variation with k and s_y of the initial slope of the η vs. ε curve when buckling occurs — $s_x = 0$.

instant of buckling. The corresponding decrease in the buckling load is, of course, sufficient to satisfy the necessary condition that the total work done in applying a given end strain is reduced by the shear deformability.

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(*Received* 5 *December* 1966; *revised* 18 *August* 1967)

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